CS 543: Computer Graphics
Lecture 3 (Part I): Fractals

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What are Fractals?

- Mathematical expressions
- Approach infinity in organized way
- Utilizes recursion on computers
- Popularized by Benoit Mandelbrot (Yale university)
- Dimensional:
  - Line is one-dimensional
  - Plane is two-dimensional
- Defined in terms of self-similarity

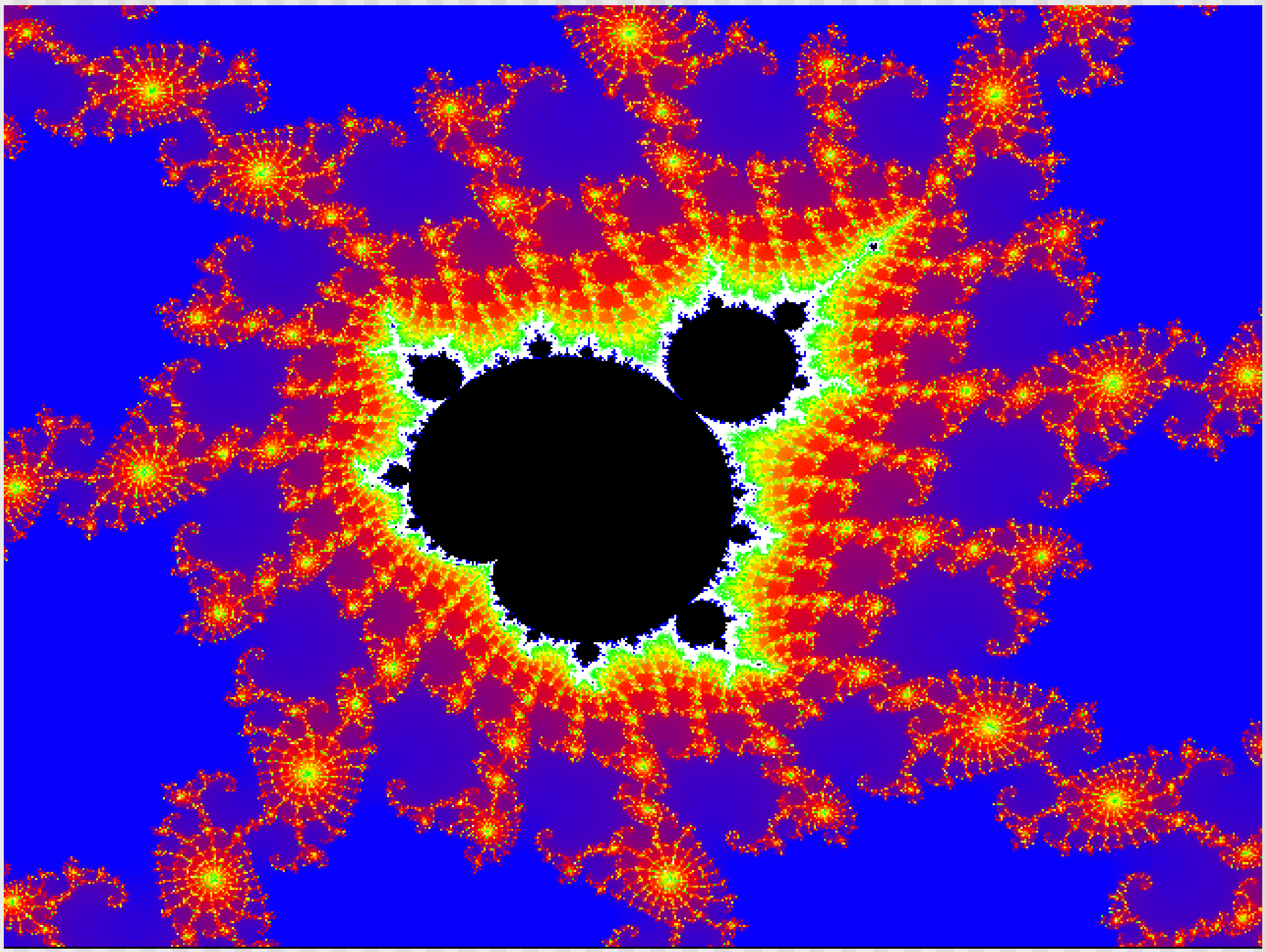
Fractals: Self-similarity

- Level of detail remains the same as we zoom in
- Example: surface roughness or profile same as we zoom in
- Types:
  - Exactly self-similar
  - Statistically self-similar

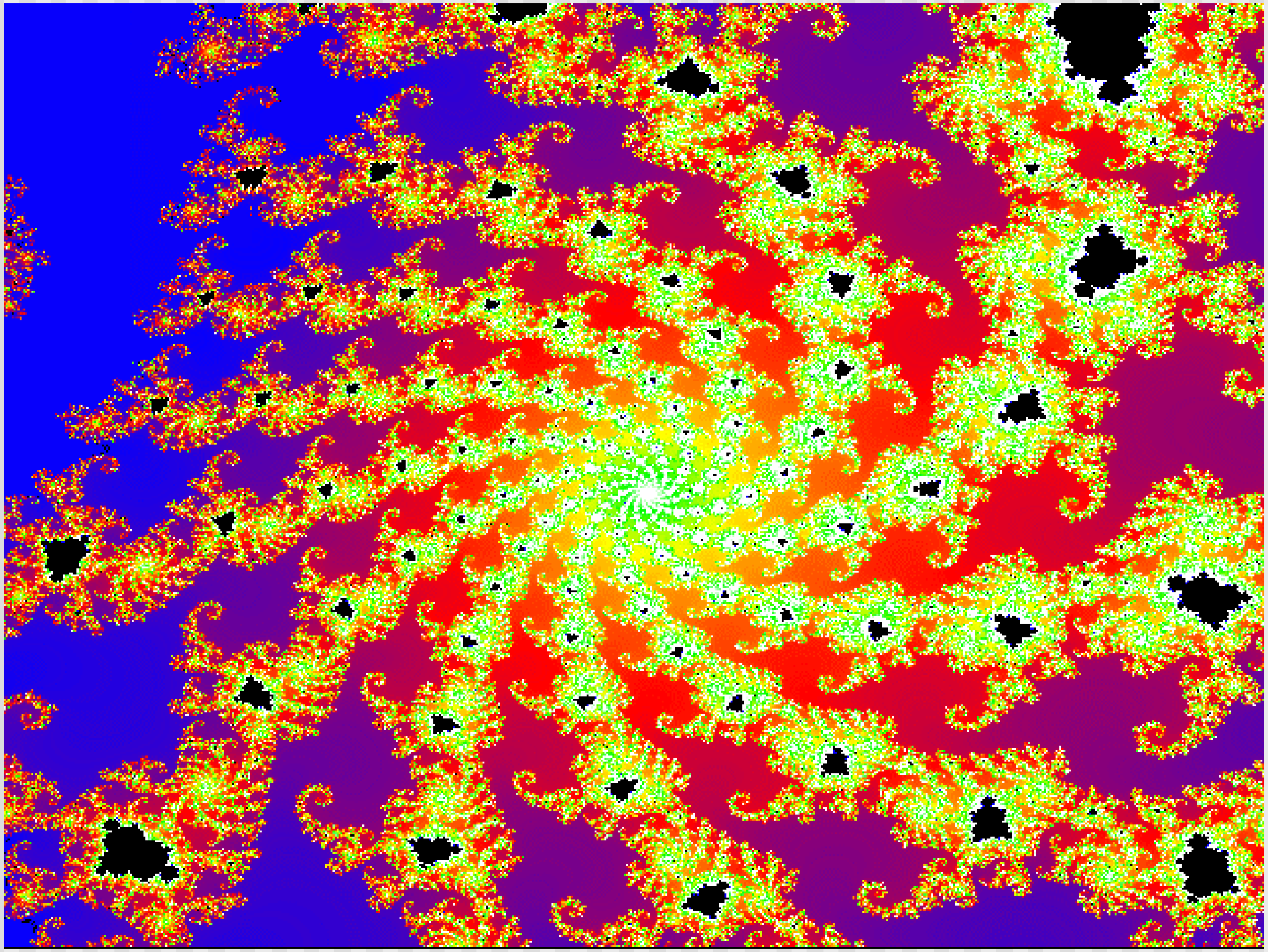
Examples of Fractals

- Clouds
- Grass
- Fire
- Modeling mountains (terrain)
- Coastline
- Branches of a tree
- Surface of a sponge
- Cracks in the pavement
- Designing antennae (www.fractenna.com)

Example: Mandelbrot Set



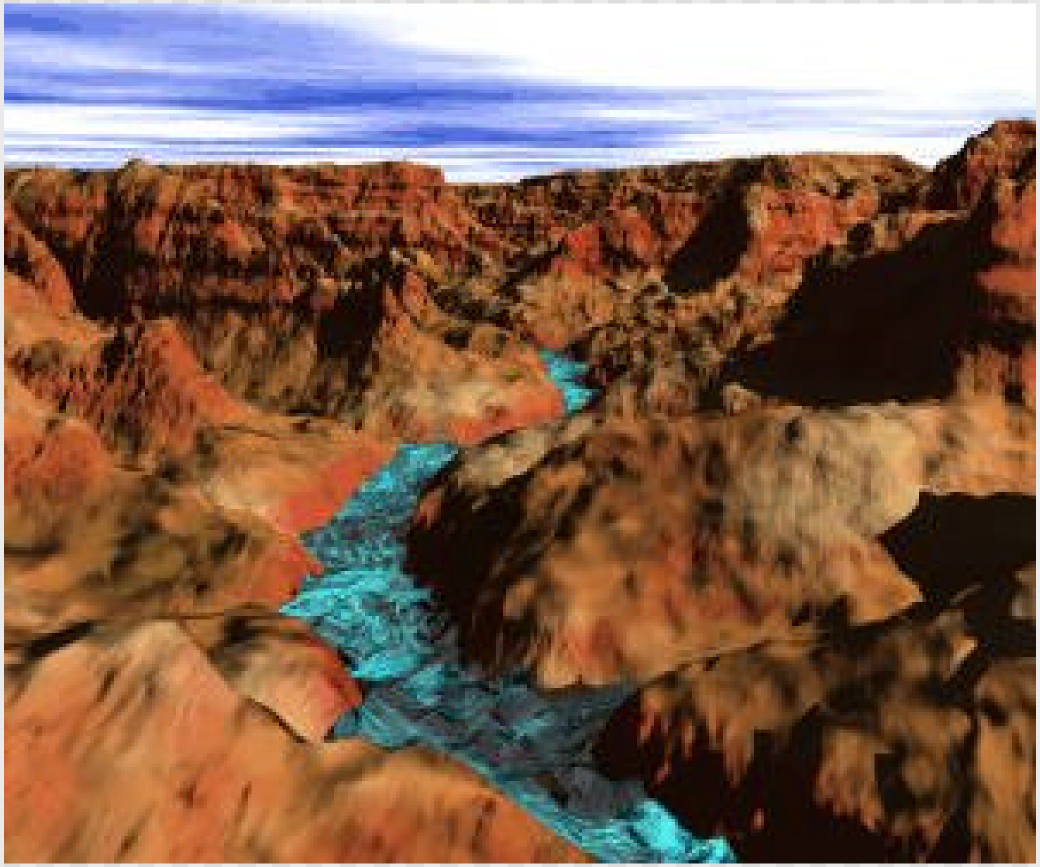
Example: Mandelbrot Set



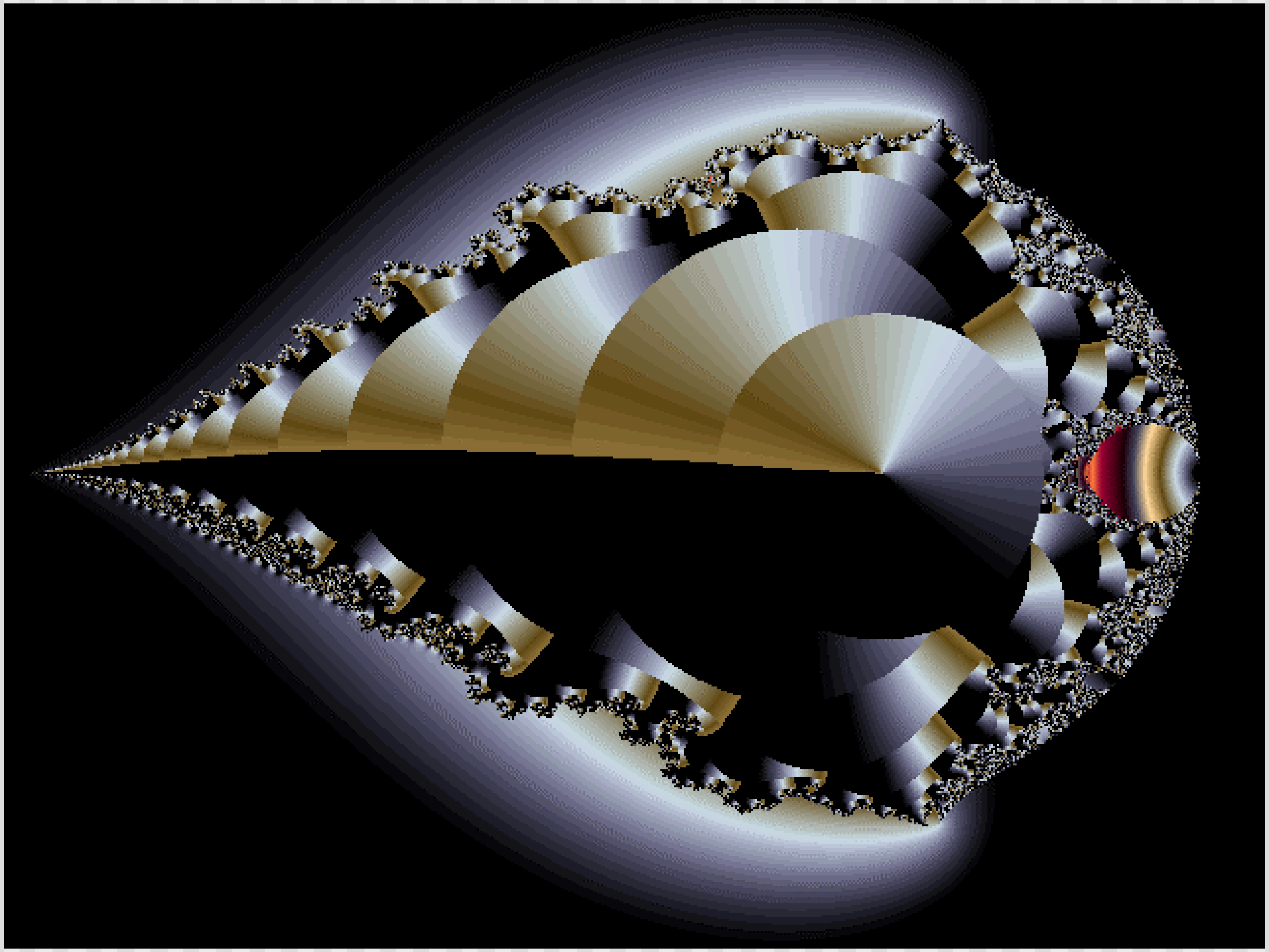
Example: Fractal Terrain

Courtesy: Mountain 3D Fractal Terrain software

Example: Fractal Terrain

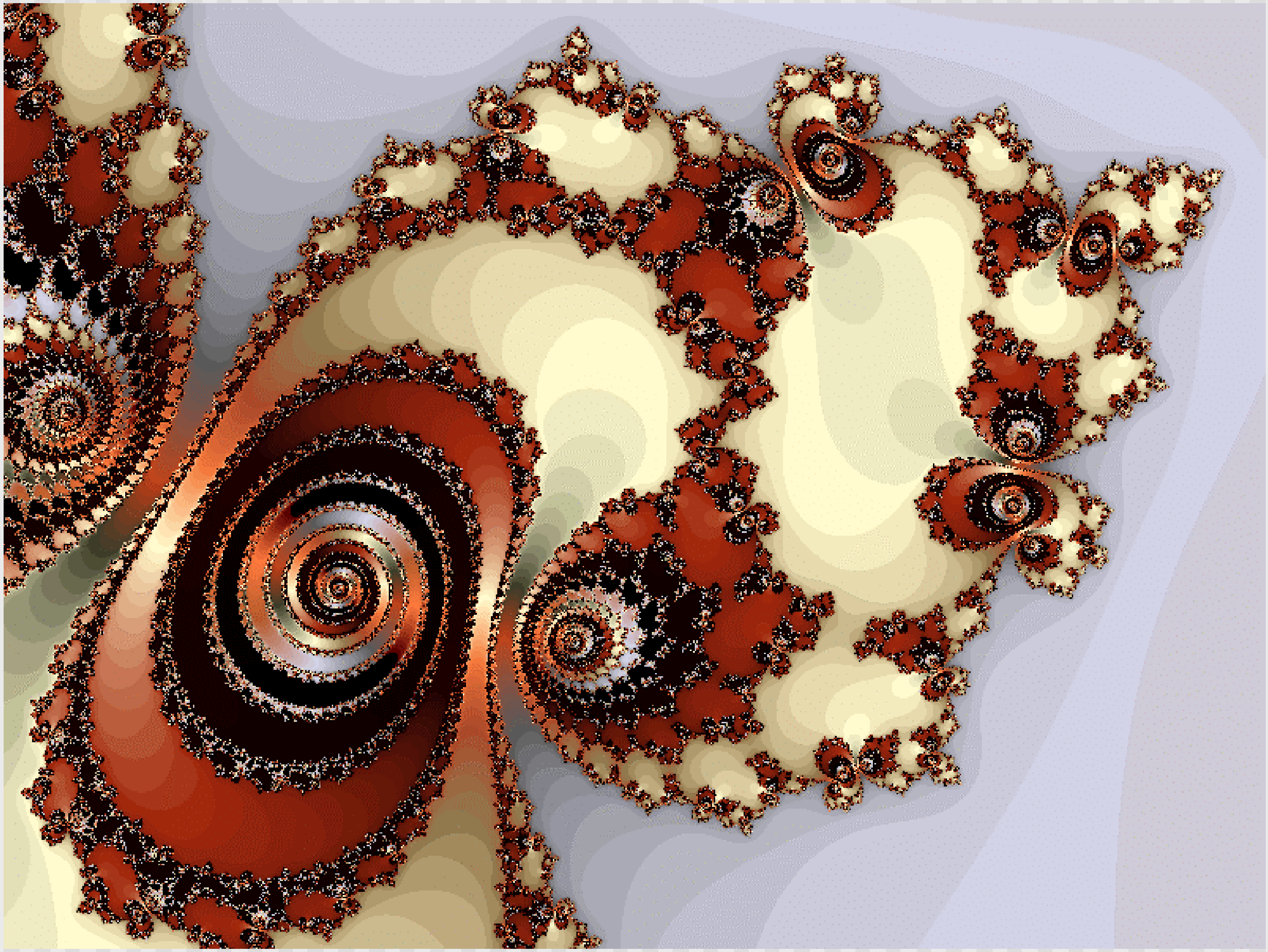


Example: Fractal Art



Courtesy: Internet Fractal Art Contest

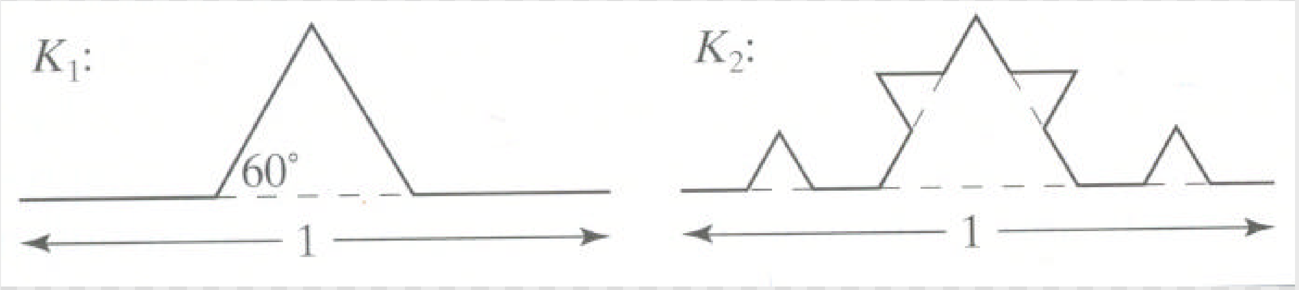
Application: Fractal Art



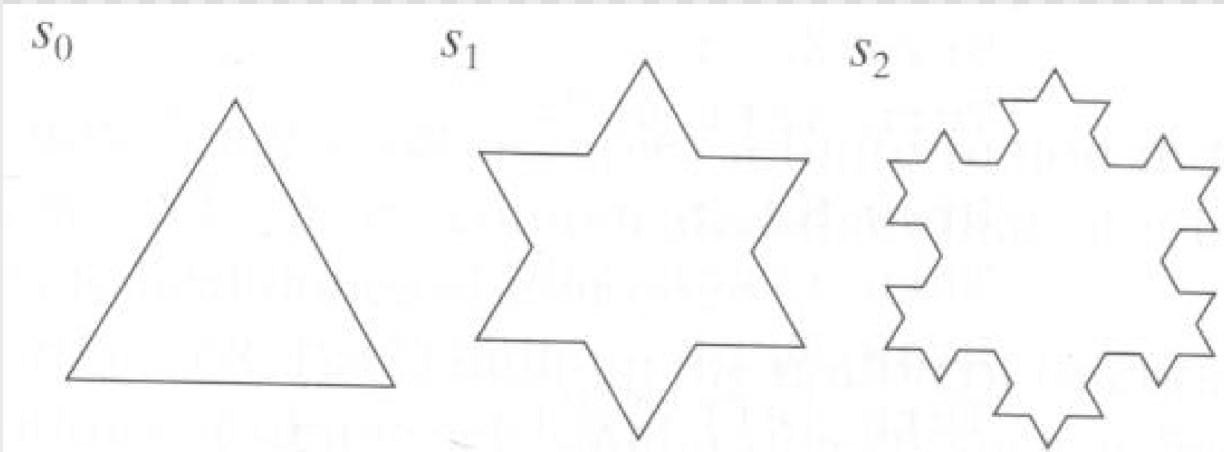
Courtesy: Internet Fractal Art Contest

Koch Curves

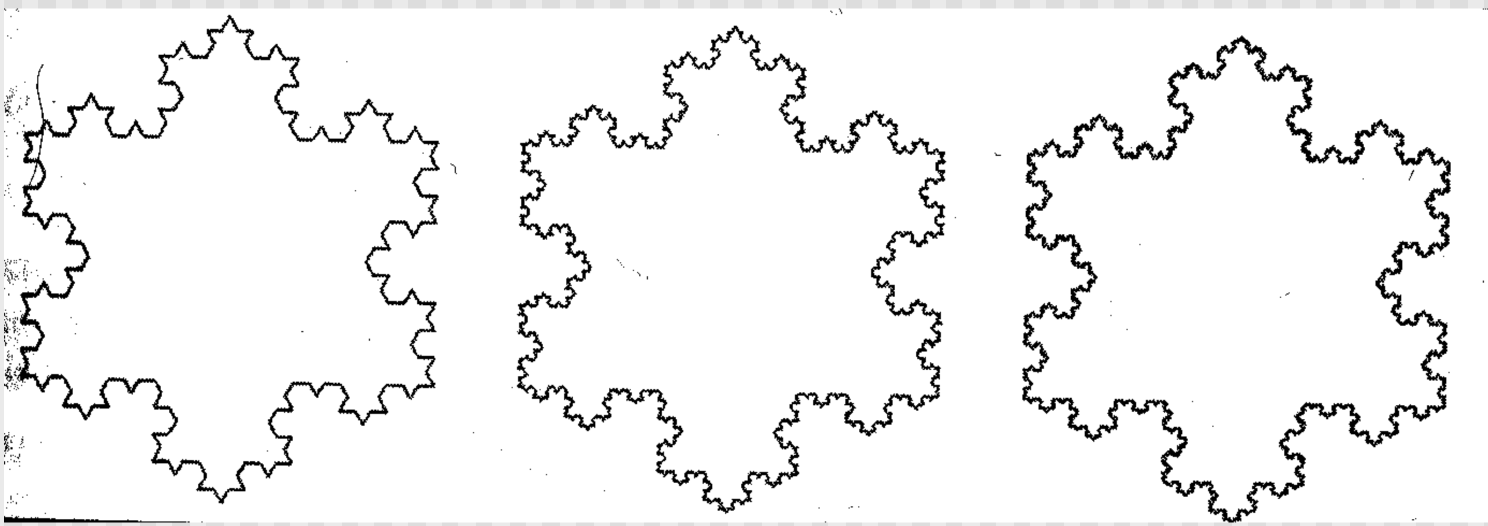
- Discovered in 1904 by Helge von Koch
- Start with straight line of length 1
- Recursively:
  - Divide line into 3 equal parts
  - Replace middle section with triangular bump with sides of length 1/3
  - New length = 4/3



Koch Curves



S_3, S_4, S_5,



Koch Snowflakes

- Can form Koch snowflake by joining three Koch curves
- Perimeter of snowflake grows as:

$$P_i = 3\left(4/3\right)^i$$

  where P_i is the perimeter of the ith snowflake iteration
- However, area grows slowly and $S_\infty = 8/5$!!
- Self-similar:
  - zoom in on any portion
  - If *n* is large enough, shape still same
  - On computer, smallest line segment > pixel spacing

Koch Snowflakes

Pseudocode, to draw K_n:

```
If (n equals 0) draw straight line
Else{
        Draw K_{n-1}
        Turn left 60°
        Draw K_{n-1}
        Turn right 120°
        Draw K_{n-1}
        Turn left 60°
        Draw K_{n-1}
}
```

Iterated Function Systems (IFS)

- Recursively call a function
- Does result converge to an image? What image?
- IFS's converge to an image
- Examples:
  - The Fern
  - The Mandelbrot set

The Fern

Mandelbrot Set

- Based on iteration theory
- Function of interest:

$$f(z) = (s)^2 + c$$

- Sequence of values (or orbit):

$$d_1 = (s)^2 + c$$

$$d_2 = ((s)^2 + c)^2 + c$$

$$d_3 = (((s)^2 + c)^2 + c)^2 + c$$

$$d_4 = ((((s)^2 + c)^2 + c)^2 + c)^2 + c$$

Mandelbrot Set

- Orbit depends on *s* and *c*
- Basic question,:
  - For given *s* and *c,*
    - does function stay finite? (within Mandelbrot set)
    - explode to infinity? (outside Mandelbrot set)
- Definition: if |d| < 1, orbit is finite else inifinite
- Examples orbits:
  - *s* = 0, *c* = -1, orbit = 0,-1,0,-1,0,-1,0,-1,…..*finite*
  - *s* = 0, *c* = 1, orbit = 0,1,2,5,26,677…… *explodes*

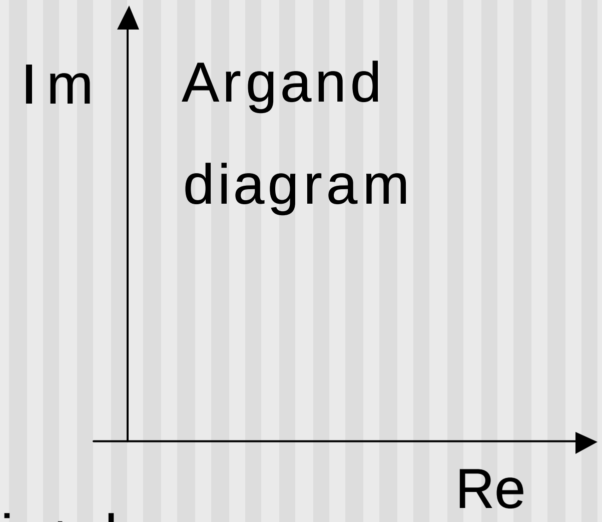
Mandelbrot Set

- Mandelbrot set: use complex numbers for *c* and *s*
- Always set $s = 0$
- Choose c as a complex number
- For example:
    - $s = 0$, $c = 0.2 + 0.5i$
- Hence, orbit:
    - $0, \quad c, \quad c^2, \quad c^2 + c, \quad (c^2 + c)^2 + c, \ldots\ldots\ldots$
- Definition: Mandelbrot set includes all finite orbit *c*

Mandelbrot Set

- Some complex number math:

$$i * i = -1$$



- For example:

$$2i * 3i = -6$$

- Modulus of a complex number, z = ai + b:

$$|z| = \sqrt{a^2 + b^2}$$

- Squaring a complex number:

$$x + yi = (x^2 - y^2) + (2xy)i$$

- Ref: Hill Appendix 2.3

Mandelbrot Set

- Calculate first 4 terms
  - with s=2, c=-1
  - with s = 0, c = -2+i

Mandelbrot Set

- Calculate first 3 terms
  - with s=2, c=-1, terms are

$$2^2-1=3$$
$$3^2-1=8$$
$$8^2-1=63$$

  - with s = 0, c = -2+i

$$0+(-2+i)=-2+i$$
$$(-2+i)^2+(-2+i)=1-3i$$
$$(1-3i)^2+(-2+i)=-10-5i$$

Mandelbrot Set

- **Fixed points:** Some complex numbers converge to certain values after *x* iterations.
- **Example:**
  - s = 0, *c* = -0.2 + 0.5i converges to –0.249227 + 0.333677i after 80 iterations
  - **Experiment:** square –0.249227 + 0.333677i and add -0.2 + 0.5i
- Mandelbrot set depends on the fact the convergence of certain complex numbers

Mandelbrot Set

- Routine to draw Mandelbrot set:
- Cannot iterate forever: our program will hang!
- Instead iterate 100 times
- Math theorem:
  - if number hasn't exceeded 2 after 100 iterations, never will!
- Routine returns:
  - Number of times iterated before modulus exceeds 2, *or*
  - 100, if modulus doesn't exceed 2 after 100 iterations
  - See dwell( ) function in Hill (figure 9.49, pg. 510)

Mandelbrot dwell( ) function (pg. 510)

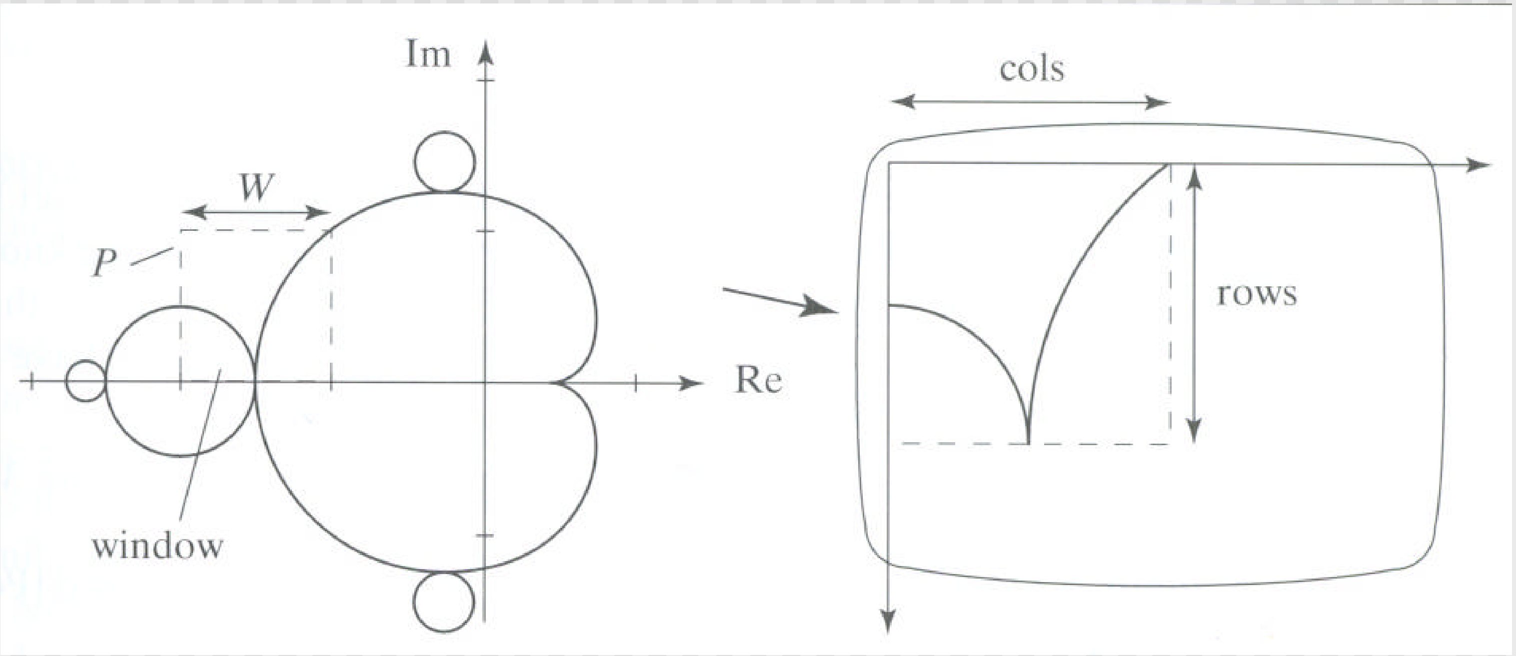
```
int dwell(double cx, double cy)
{ // return true dwell or Num, whichever is smaller
   #define Num 100 // increase this for better pics

   double tmp, dx = cx, dy = cy, fsq = cx*cx + cy*cy;
   for(int count = 0;count <= Num && fsq <= 4; count++)
   {
       tmp = dx;     // save old real part
       dx = dx*dx – dy*dy + cx; // new real part
       dy = 2.0 * tmp * dy + cy; // new imag. Part
       fsq = dx*dx + dy*dy;
   }
   return count; // number of iterations used
}
```

Mandelbrot Set

- Map real part to x-axis
- Map imaginary part to y-axis
- Set world window to range of complex numbers to investigate. E.g:
  - X in range [-2.25: 0.75]
  - Y in range [-1.5: 1.5]
- Choose your viewport. E.g:
  - Viewport = [V.L, V.R, V.B, V.T]= [60,380,80,240]
- Do window-to-viewport mapping

Mandelbrot Set

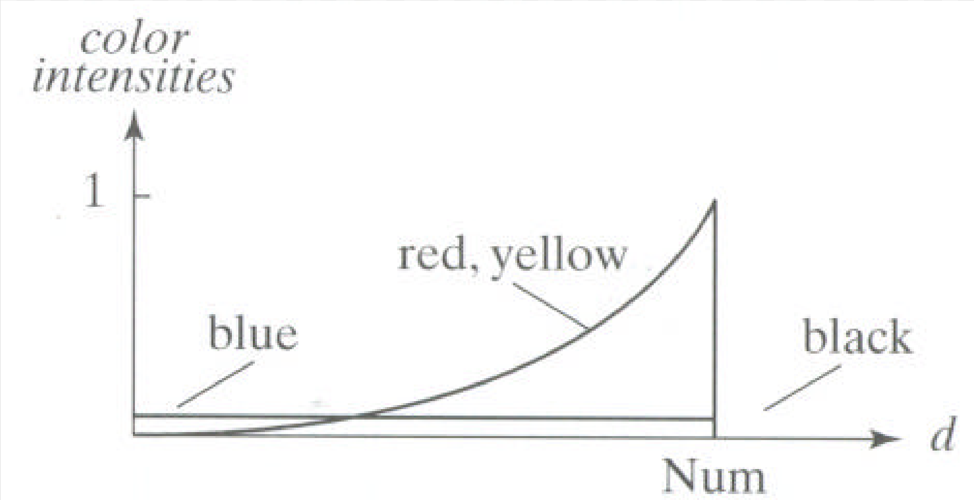


Mandelbrot Set

- So, for each pixel:
  - Compute corresponding point in world
  - Call your dwell( ) function
  - Assign color <Red,Green,Blue> based on dwell( ) return value
- Choice of color determines how pretty
- Color assignment:
  - Basic: In set (i.e. dwell( ) = 100), color = black, else color = white
  - Discrete: Ranges of return values map to same color
    - E.g 0 – 20 iterations = color 1
    - 20 – 40 iterations = color 2, etc.
  - Continuous: Use a function

Mandelbrot Set

Use continuous function



FREE SOFTWARE

- Free fractal generating software
  - Fractint
  - FracZoom
  - Astro Fractals
  - Fractal Studio
  - 3DFract

References

- Hill, chapter 9